

Inverse Scattering Problem for Vector Fields and the Heavenly Equation

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Abstract

We solve the inverse scattering problem for multidimensional vector fields and we use this result to construct the formal solution of the Cauchy problem for the second heavenly equation of Plebanski, a scalar nonlinear partial differential equation in four dimensions relevant in General Relativity, which arises from the commutation of multidimensional Hamiltonian vector fields.

1 Introduction

In this paper we solve the inverse scattering problem for multidimensional vector fields and we use this result to construct the formal solution of the Cauchy problem for the real second heavenly equation

$$\theta_{,tx} - \theta_{,zy} + \theta_{,xx}\theta_{,yy} - \theta_{,xy}^2 = 0, \quad \theta = \theta(x, y, z, t) \in \mathbb{R}, \quad x, y, z, t \in \mathbb{R}, \quad (1)$$

where $\theta_{,x} = \partial\theta/\partial x$, $\theta_{,xy} = \partial^2\theta/\partial x\partial y$.

This scalar nonlinear partial differential equation (PDE) in 4 independent variables x, y, z, t , introduced in [1] by Plebanski, describes the Einstein field

equations that govern self-dual gravitational fields. Its 2-dimensional reduction $\theta_{,z} = \theta_{,t} = 0$ is the Monge-Ampère equation, relevant in Differential Geometry. As we shall see in the following, the heavenly equation plays also a distinguished role in the theory of commuting 2-dimensional, Hamiltonian dynamical systems.

The heavenly equation, together with the equations for the self-dual (and anti-self-dual) Yang-Mills (SDYM) fields [2], are perhaps the most distinguished examples of nonlinear PDEs in more than three independent variables arising as commutativity conditions of linear operators [3], and therefore amenable, in principle, to exact treatments based on the spectral theory of those operators [4],[5]. If the SDYM equations are considered on an abstract Lie algebra, then the heavenly equation can actually be interpreted as a distinguished realization of the SDYM equations, corresponding to the Lie algebra of divergence free vector fields independent of the SDYM coordinates [6]. Equation (1) has been investigated within the twistor approach in [7]. A bi-Hamiltonian formulation, a hodograph transformation, and a nonlinear dressing formalism for equation (1) have been recently constructed, respectively, in [8], [9] and [10].

We end this introductory section remarking that the inverse scattering problem for vector fields developed in this paper presents some interesting new features with respect to the traditional Inverse Scattering Transform (IST) in multidimensions. For instance:

- The space of eigenfunctions of the multidimensional vector fields is a ring. In the heavenly reduction, the space of eigenfunctions is not only a ring, but also a Lie algebra, with Lie bracket given by the natural Poisson bracket.
- Constructing solutions of the integrable multidimensional PDEs arising from the commutation of vector fields is equivalent to characterizing commuting pairs of dynamical systems, Hamiltonian in the heavenly reduction.
- The scattering data are intimately related to the scattering vector $\vec{\Delta}$, associated with the dynamical system which describes the characteristic curves of the vector field. These scattering data, in one to one correspondence with the coefficients of the vector field, depend on a number of spectral variables equal to the number of space variables.

Thus the counting is consistent (a highly non generic feature in multi-dimensional IST).

- A central role in this IST is played by a certain Riemann-Hilbert problem with a shift depending on the scattering vector $\vec{\Delta}$.
- The above ring property allows one to construct two essentially different IST schemes. The first one, in terms of a set of eigenfunctions intimately related to the integral curves of the associated vector field, is characterized by a nonlinear inverse problem. The second one, involving more traditional Jost and analytic eigenfunctions, obtained “exponentiating” the above eigenfunctions, is instead characterized by a linear inverse problem.

2 The Cauchy problem

A multidimensional Lax pair. It is known that the commutation of multidimensional vector fields leads to nonlinear first order multidimensional PDEs (see, f.i., [3]).

Consider, for instance, the pair of operators

$$\hat{L}_i = \partial_{t^i} + \lambda \partial_{z^i} + \sum_{k=1}^N u_i^k \partial_{x^k} = \partial_{t^i} + \lambda \partial_{z^i} + \vec{u}_i \cdot \nabla_{\vec{x}}, \quad i = 1, 2 \quad (2)$$

where ∂_x denotes partial differentiation with respect to the generic variable x , $\vec{x} = (x^1, \dots, x^N)$, $\nabla_{\vec{x}} = (\partial_{x^1}, \dots, \partial_{x^N})$, $\vec{u}_i = (u_i^1, \dots, u_i^N)$, $i = 1, 2$, λ is a complex parameter and the vector coefficients \vec{u}_i depend on the independent variables t^i, z^i, x^k , $i = 1, 2$, $k = 1, \dots, N$, but not on λ . The existence of a common eigenfunction f for the operators \hat{L}_1 and \hat{L}_2 :

$$\hat{L}_1 f = \hat{L}_2 f = 0, \quad (3)$$

implies their commutation, $\forall \lambda$:

$$[\hat{L}_1, \hat{L}_2] = 0, \quad (4)$$

which is equivalent to the following system of $2N$ first order quasi-linear PDEs in $(4 + N)$ dimensions:

$$\begin{aligned} u_{1,z^2}^k &= u_{2,z^1}^k, & k &= 1, \dots, N, \\ u_{1,t^2}^k - u_{2,t^1}^k + \sum_{l=1}^N \left(u_2^l u_{1,x^l}^k - u_1^l u_{2,x^l}^k \right) &= 0, & k &= 1, \dots, N. \end{aligned} \quad (5)$$

Parametrizing the first set of equations in terms of the potentials U^k

$$u_i^k = U_{,z^i}^k, \quad i = 1, 2, \quad k = 1, \dots, N, \quad (6)$$

one obtains the following system of N nonlinear PDEs for the N dependent variables U^k in $(4 + N)$ dimensions:

$$U_{,t^1 z^2}^k - U_{,t^2 z^1}^k + \sum_{l=1}^N (U_{,z^1}^l U_{,z^2 x^l}^k - U_{,z^2}^l U_{,z^1 x^l}^k) = 0, \quad k = 1, \dots, N. \quad (7)$$

This system admits a natural reduction; indeed, applying the operator $\sum_{k=1}^N \partial_{x^k}$ to equations (7) one obtains

$$\left[\partial_{t^1} \partial_{z^2} - \partial_{t^2} \partial_{z^1} + \sum_{k=1}^N (U_{,z^1}^k \partial_{z^2} \partial_{x^k} - U_{,z^2}^k \partial_{z^1} \partial_{x^k}) \right] \sum_{l=1}^N U_{,x^l}^l = 0, \quad (8)$$

from which one infers that the condition

$$\sum_{k=1}^N U_{,x^k}^k = 0 \quad (9)$$

is an admissible reduction for equation (7), implying that the condition of zero-divergence:

$$\nabla_{\vec{x}} \cdot \vec{u}_i = 0, \quad i = 1, 2 \quad (10)$$

is an admissible constraint for the vectors \vec{u}_i , $i = 1, 2$.

From now on, we concentrate our attention on the following important example:

$$N = 2, \quad z^i = x^i, \quad i = 1, 2. \quad (11)$$

In this case, the zero-divergence reduction (10) makes the two vector fields $\vec{u}_i \cdot \nabla_{\vec{x}}$ Hamiltonian, allowing for the introduction of two Hamiltonians H_i , $i = 1, 2$ such that:

$$u_i^j = \epsilon^{jk} H_{i,x^k}, \quad i, j, k = 1, 2, \quad (12)$$

where ϵ^{jk} is the totally anti-symmetric tensor, which, due to (5a), are parametrized by a single potential θ :

$$H_i = \theta_{,x^i} \quad (u_i^j = \epsilon^{jk} H_{i,x^k} = \epsilon^{jk} \theta_{,x^i x^k}, \quad U^i = \epsilon^{jk} H_j = \epsilon^{ij} \theta_{,x^j}, \quad i, j = 1, 2). \quad (13)$$

Then the compatible linear problems (3) can be written down as Hamilton equations with respect to the times t^1, t^2 :

$$\begin{aligned} f_{,t^1} &= \{H_1 + \lambda x^2, f\}_{\vec{x}}, \\ f_{,t^2} &= \{H_2 - \lambda x^1, f\}_{\vec{x}}, \end{aligned} \quad (14)$$

where $\{\cdot, \cdot\}_{\vec{x}}$ is the Poisson bracket with respect to the variables x^1, x^2 :

$$\{f, g\}_{\vec{x}} = f_{,x^1} g_{,x^2} - f_{,x^2} g_{,x^1}, \quad (15)$$

and the nonlinear system (7) reduces to the heavenly equation in Hamiltonian form

$$\theta_{,t^2 x^1} - \theta_{,t^1 x^2} + \{\theta_{,x^1}, \theta_{,x^2}\}_{\vec{x}} = \text{constant}, \quad (16)$$

equivalent to (1) after choosing the constant to be zero, with the following identification of the variables

$$t^1 = z, \quad t^2 = t, \quad x^1 = x, \quad x^2 = y, \quad (17)$$

which we are going to make in the rest of the paper.

Commuting dynamical systems. It is well-known (see, f.i., [11]) that linear first order PDEs like (3) are intimately related to systems of ordinary differential equations describing their characteristic curves. The dynamical systems associated with the vector fields $\hat{L}_{1,2}$:

$$\begin{aligned} \hat{L}_1 &= \partial_z + \lambda \partial_x + \vec{u}_1 \cdot \nabla_{\vec{x}}, \\ \hat{L}_2 &= \partial_t + \lambda \partial_y + \vec{u}_2 \cdot \nabla_{\vec{x}}, \end{aligned} \quad (18)$$

where $\vec{x} = (x, y)$ and $\nabla_{\vec{x}} = (\partial_x, \partial_y)$, are:

$$\begin{aligned} \hat{L}_1 : \quad \frac{d\vec{x}}{dz} &= \vec{u}_1(\vec{x}, z) + (\lambda, 0), \\ \hat{L}_2 : \quad \frac{d\vec{x}}{dt} &= \vec{u}_2(\vec{x}, t) + (0, \lambda). \end{aligned} \quad (19)$$

We remark that the two flows generated by the times z and t commute, $\forall \lambda$, iff the fields u_i^j satisfy the integrable quasi-linear equations (5),(11). Therefore:

Constructing solutions of the integrable PDEs (5),(11) is equivalent to solving the classical problem of constructing pairs of commuting 2-dimensional dynamical systems.

We also remark that, in the heavenly zero-divergence reduction (10), the commuting flows (19) are Hamiltonian:

$$\begin{aligned}\frac{d\vec{x}}{dz} &= \{\vec{x}, H_1 + \lambda y\}_{\vec{x}}, \\ \frac{d\vec{x}}{dt} &= \{\vec{x}, H_2 - \lambda x\}_{\vec{x}}.\end{aligned}\tag{20}$$

Consider now a solution (r^1, r^2) of equation (19a) and assume that $u_1^j \rightarrow 0$ as $|z| \rightarrow \infty$; then the phase space point $\vec{r} = (r^1, r^2)$ travels asymptotically with constant speed $(\lambda, 0)$:

$$\vec{r} \rightarrow \vec{s}_{\pm} + (\lambda, 0)z, \quad z \rightarrow \pm\infty,\tag{21}$$

where \vec{s}_{\pm} are constant vectors.

If \vec{s}_- is given, then \vec{s}_+ can be viewed as a function of \vec{s}_- and λ : $\vec{s}_+(\vec{s}_-, \lambda)$, and the difference $\vec{\Delta}$ between these asymptotic positions

$$\vec{\Delta}(\vec{s}_-, \lambda) = \vec{s}_+ - \vec{s}_-\tag{22}$$

describes the z -scattering of the phase space point of the dynamical system (19a). As we shall see in the following, the scattering vector $\vec{\Delta}$ is also intimately related to the z -scattering datum S associated with the partial differential operator \hat{L}_1 (see (51)), which plays a central role in the IST for the vector field \hat{L}_1 , and, consequently, in the solution of the Cauchy problem for the nonlinear PDEs (5),(11), and for the heavenly reduction (1).

Now we consider such a Cauchy problem, within the class of rapidly decreasing real potentials u_i^j :

$$\begin{aligned}u_i^j &\rightarrow 0, \quad (x^2 + y^2 + z^2) \rightarrow \infty, \\ u_i^j &\in \mathbb{R}, \quad (x, y, z) \in \mathbb{R}^3, \quad t > 0,\end{aligned}\tag{23}$$

interpreting t as time and the other three variables x, y, z as space variables.

To solve such a Cauchy problem by the IST method (see, f.i., [4],[5]), we construct the IST for the operator \hat{L}_1 in (18a), within the class of rapidly decreasing real potentials, interpreting the operator \hat{L}_2 in (18b) as the time operator.

Eigenfunctions and their properties. Since $\hat{L}_{1,2}$ are linear, first order, partial differential operators with scalar coefficients, the product of two solutions $f_{1,2}$ of the equation $\hat{L}_i f = 0$ is also a solution: $\hat{L}_i(f_1 f_2) = 0$ (in

general, an arbitrary function $G(f_1, f_2)$ of these two solutions is also a solution: $\hat{L}_i G(f_1, f_2) = 0$. Therefore the space of solutions of the equation $\hat{L}_i f = 0$ forms a ring. As we shall see in the following, this property introduces important novelties in the Inverse Scattering formalism.

The localization (23) of the vector potential \vec{u}_1 implies that, if f is a solution of $\hat{L}_1 f = 0$, then

$$\begin{aligned} f(\vec{x}, z, \lambda) &\rightarrow f_{\pm}(\vec{X}, \lambda), \quad z \rightarrow \pm\infty, \\ \vec{X} &:= \vec{x} - (\lambda, 0)z; \end{aligned} \quad (24)$$

i.e., asymptotically, f is an arbitrary function of $(x - \lambda z)$, y and λ .

A central role in the theory is played by the real vector eigenfunctions $\vec{\varphi}_{\pm}(\vec{x}, z, \lambda)$, the solutions of $\hat{L}_1 \vec{\varphi}_{\pm} = \vec{0}$ defined by the asymptotics

$$\vec{\varphi}_{\pm}(\vec{x}, z, \lambda) \rightarrow \vec{X}, \quad z \rightarrow \pm\infty. \quad (25)$$

Their connection to the dynamical system (19a) is immediate. Let $\vec{x} = \vec{r}_{\pm}(\vec{\omega}, z, \lambda)$ be the solutions of the dynamical system (19a) satisfying:

$$\begin{aligned} \frac{d\vec{r}_{\pm}}{dz} &= \vec{u}_1(\vec{r}_{\pm}, z) + (\lambda, 0), \\ \vec{r}_{\pm}(\vec{\omega}, z, \lambda) &\rightarrow \vec{\omega} + (\lambda, 0)z, \quad z \rightarrow \pm\infty, \end{aligned} \quad (26)$$

where $\vec{\omega} \in \mathbb{R}^2$ is a given constant. Then the vector eigenfunctions $\vec{\varphi}_{\pm}$ arise from solving the equation $\vec{x} = \vec{r}_{\pm}(\vec{\omega}, z, \lambda)$ with respect to $\vec{\omega}$:

$$\vec{x} = \vec{r}_{\pm}(\vec{\omega}, z, \lambda) \Leftrightarrow \vec{\omega} = \vec{\varphi}_{\pm}(\vec{x}, z, \lambda). \quad (27)$$

Hereafter, when systems of equations are inverted, we will always consider a neighborhood of a point in which the corresponding Jacobian does not vanish.

Due to the ring property of the space of eigenfunctions, an arbitrary function of $\vec{\varphi}_{\pm}$ is also an eigenfunction. The Jost eigenfunctions ϕ^{\pm} are defined by:

$$\phi^{\pm}(\vec{x}, z; \vec{\alpha}, \lambda) := e^{i\vec{\alpha} \cdot \vec{\varphi}_{\pm}(\vec{x}, z, \lambda)}. \quad (28)$$

They satisfy the asymptotics:

$$\phi^{\pm}(\vec{x}, z; \vec{\alpha}, \lambda) \rightarrow e^{i\vec{\alpha} \cdot \vec{X}}, \quad z \rightarrow \pm\infty, \quad (29)$$

where $\vec{\alpha} = (\alpha_1, \alpha_2)$ are arbitrary real parameters and $\vec{\alpha} \cdot \vec{X}$ is the usual scalar product in \mathbb{R}^2 . They are equivalently characterized by the integral equations

$$\phi^\pm(\vec{x}, z; \vec{\alpha}, \lambda) + \int_{\mathbb{R}^3} d\vec{x}' dz' G^\pm(\vec{x} - \vec{x}', z - z'; \lambda) (\vec{u}_1(\vec{x}', z') \cdot \nabla_{\vec{x}'}) \phi^\pm(\vec{x}', z'; \vec{\alpha}, \lambda) = e^{i\vec{\alpha} \cdot \vec{X}}, \quad (30)$$

in terms of the Green's functions

$$G^\pm(\vec{x}, z; \lambda) = \mp \theta(\mp z) \delta(x - \lambda z) \delta(y). \quad (31)$$

A crucial role in inverse scattering is also played by analytic eigenfunctions. In the IST for the vector field \hat{L}_1 , the analytic eigenfunctions $\psi_\pm(\vec{x}, z; \vec{\alpha}, \lambda)$ are the solutions of $\hat{L}_1 \psi_\pm = 0$ satisfying the integral equations

$$\psi_\pm(\vec{x}, z; \vec{\alpha}, \lambda) + \int_{\mathbb{R}^3} d\vec{x}' dz' G_\pm(\vec{x} - \vec{x}', z - z'; \lambda) e^{i\vec{\alpha} \cdot (\vec{X} - \vec{X}')} (\vec{u}_1(\vec{x}', z') \cdot \nabla_{\vec{x}'}) \psi_\pm(\vec{x}', z'; \vec{\alpha}, \lambda) = e^{i\vec{\alpha} \cdot \vec{X}}, \quad (32)$$

where G_\pm are the analytic Green's functions

$$G_\pm(\vec{x}, z; \lambda) = \pm \frac{\delta(y)}{2\pi i [x - (\lambda \pm i\epsilon)z]}. \quad (33)$$

The analyticity properties of $G_\pm(\vec{x}, z; \lambda)$ in the complex λ -plane imply that $\psi_+(\vec{x}, z; \vec{\alpha}, \lambda) e^{-i\vec{\alpha} \cdot \vec{X}}$ and $\psi_-(\vec{x}, z; \vec{\alpha}, \lambda) e^{-i\vec{\alpha} \cdot \vec{X}}$ are analytic, respectively, in the upper and lower half λ - plane, with the following asymptotics, for large λ :

$$\psi_\pm(\vec{x}, z; \vec{\alpha}, \lambda) e^{-i\vec{\alpha} \cdot \vec{X}} = 1 + \frac{\vec{\alpha} \cdot \vec{Q}_\pm(\vec{x}, z)}{\lambda} + O(\lambda^{-2}), \quad |\lambda| \gg 1, \quad (34)$$

where:

$$\vec{Q}_\pm(\vec{x}, z) = \pm P \int_{\mathbb{R}^2} \frac{dx' dz'}{2\pi(z - z')} \vec{u}_1(x', y, z') - \frac{i}{2} \left(\int_{-\infty}^x - \int_x^\infty \right) dx' \vec{u}_1(x', y, z), \quad (35)$$

entailing that

$$\vec{u}_1(\vec{x}, z) = i \vec{Q}_{\pm, x}(\vec{x}, z). \quad (36)$$

It is important to remark that the analytic Green's functions (33) exhibit the following asymptotics for $z \rightarrow \pm\infty$:

$$\begin{aligned} G_\pm(\vec{x}, z; \lambda) &\rightarrow \pm \frac{\delta(y)}{2\pi i [x - \lambda z \mp i\epsilon]}, \quad z \rightarrow +\infty, \\ G_\pm(\vec{x}, z; \lambda) &\rightarrow \pm \frac{\delta(y)}{2\pi i [x - \lambda z \pm i\epsilon]}, \quad z \rightarrow -\infty, \end{aligned} \quad (37)$$

entailing that the $z = +\infty$ asymptotics of ψ_+ and ψ_- are analytic respectively in the lower and upper halves of the complex plane $(x - \lambda z)$, while the $z = -\infty$ asymptotics of ψ_+ and ψ_- are analytic respectively in the upper and lower halves of the complex plane $(x - \lambda z)$ (similar features were obtained in [12]).

Since the solution space of the equation $\hat{L}_1 f = 0$ is a ring, then the product of two Jost (analytic) solutions corresponding to different vector parameters $\vec{\alpha}, \vec{\beta} \in \mathbb{R}^2$ is still a Jost (analytic) solution satisfying:

$$\begin{aligned}\phi^\pm(\vec{x}, z; \vec{\alpha}, \lambda) \phi^\pm(\vec{x}, z; \vec{\beta}, \lambda) &= \phi^\pm(\vec{x}, z; \vec{\alpha} + \vec{\beta}, \lambda), \\ \psi_\pm(\vec{x}, z; \vec{\alpha}, \lambda) \psi_\pm(\vec{x}, z; \vec{\beta}, \lambda) &= \psi_\pm(\vec{x}, z; \vec{\alpha} + \vec{\beta}, \lambda).\end{aligned}\tag{38}$$

Choosing $\vec{\alpha} = (1, 0)$ and $\vec{\beta} = (0, 1)$, equations (38) imply that the Jost (analytic) eigenfunctions depend on $\vec{\alpha}$ only at the exponents:

$$\begin{aligned}\phi^\pm(\vec{x}, z; \vec{\alpha}, \lambda) &= (\phi_1^\pm(\vec{x}, z; \lambda))^{\alpha_1} (\phi_2^\pm(\vec{x}, z; \lambda))^{\alpha_2} = e^{i\vec{\alpha} \cdot \vec{\varphi}_\pm(\vec{x}, z; \lambda)}, \\ \psi_\pm(\vec{x}, z; \vec{\alpha}, \lambda) &= (\psi_{\pm 1}(\vec{x}, z; \lambda))^{\alpha_1} (\psi_{\pm 2}(\vec{x}, z; \lambda))^{\alpha_2} =: e^{\vec{\alpha} \cdot \vec{\pi}_\pm(\vec{x}, z; \lambda)},\end{aligned}\tag{39}$$

where $\psi_{\pm 1}(\vec{x}, z; \lambda)$ and $\psi_{\pm 2}(\vec{x}, z; \lambda)$ are the analytic eigenfunctions satisfying the integral equations (32) for $\vec{\alpha} = (1, 0)$ and $\vec{\alpha} = (0, 1)$ respectively.

Due to the ring property, the vector functions $\vec{\pi}_\pm$, appearing in (39b) and defined by

$$\vec{\pi}_\pm(\vec{x}, z, \lambda) := (\log \psi_{\pm 1}(\vec{x}, z; \lambda), \log \psi_{\pm 2}(\vec{x}, z; \lambda)),\tag{40}$$

are also eigenfunctions of $\hat{L}_1 : \hat{L}_1 \vec{\pi}_\pm = \vec{0}$. In addition, since ψ_\pm are analytic in $\lambda, \forall \vec{\alpha} \in \mathbb{R}^2$, it follows that $\psi_{1,2}$ in (39b) cannot have zeroes and poles in their analyticity domains. Therefore $\vec{\pi}_+$ and $\vec{\pi}_-$ are also analytic, respectively, in the upper and lower halves of the complex λ -plane, with asymptotics:

$$\vec{\pi}_\pm(\vec{x}, z, \lambda) = i\vec{X} + \frac{\vec{Q}_\pm(\vec{x}, z)}{\lambda} + O(\lambda^{-2}). \quad |\lambda| >> 1,\tag{41}$$

Analogously, the $z = +\infty$ asymptotics of $\vec{\pi}_+$ and $\vec{\pi}_-$ are analytic respectively in the lower and upper halves of the complex plane $(x - \lambda z)$, while the $z = -\infty$ asymptotics of $\vec{\pi}_+$ and $\vec{\pi}_-$ are analytic respectively in the upper and lower halves of the complex plane $(x - \lambda z)$.

Since the vector fields are real ($\vec{u}_i = \vec{u}_i^*$), we have the following obvious restrictions for the solutions ϕ^\pm and ψ_\pm (for real λ):

$$\phi^{\pm*}(\vec{x}, z; \vec{\alpha}, \lambda) = \phi^\pm(\vec{x}, z; -\vec{\alpha}, \lambda),\tag{42}$$

$$\psi_\pm^*(\vec{x}, z; \vec{\alpha}, \lambda) = \psi_\mp(\vec{x}, z; -\vec{\alpha}, \lambda).\tag{43}$$

As we shall see in the following sections, the eigenfunctions $\vec{\varphi}_\pm$ and $\vec{\pi}_\pm$ are the basic ingredients of an IST involving a nonlinear inverse problem, while the eigenfunctions ϕ^\pm and ψ_\pm are the basic ingredients of an IST involving a linear inverse problem.

Scattering data. We begin this section observing that the vectors \vec{r}_\pm defined in (26), can be identified with the vector \vec{r} in (21) setting $\vec{s}_\pm = \vec{\omega}$; then the asymptotics of \vec{r}_\pm at the opposite ends of the z axis can be described in terms of the scattering vector $\vec{\Delta}$ in (22) as follows:

$$\begin{aligned}\vec{r}_-(\vec{\omega}, z, \lambda) &\rightarrow \vec{s}_+(\vec{\omega}, \lambda) + (\lambda, 0)z, & z \rightarrow +\infty, \\ \vec{r}_+(\vec{\omega}, z, \lambda) &\rightarrow \vec{\Omega}(\vec{\omega}, \lambda) + (\lambda, 0)z, & z \rightarrow -\infty,\end{aligned}\tag{44}$$

where

$$\vec{s}_+(\vec{\omega}, \lambda) = \vec{\omega} + \vec{\Delta}(\vec{\omega}, \lambda),\tag{45}$$

and $\vec{\omega} = \vec{\Omega}(\vec{s}_+, \lambda)$ is the inverse of the transformation (45), from $\vec{\omega}$ to $\vec{s}_+(\omega, \lambda)$:

$$\vec{s}_+ = \vec{\omega} + \vec{\Delta}(\vec{\omega}, \lambda) \Leftrightarrow \vec{\omega} = \vec{\Omega}(\vec{s}_+, \lambda).\tag{46}$$

Due to the intimate connection (27) between \vec{r}_\pm and $\vec{\varphi}_\pm$, also the asymptotics of $\vec{\varphi}_\pm$ at the opposite ends of the z axis are described in terms of $\vec{\Delta}$:

$$\begin{aligned}\vec{\varphi}_-(\vec{x}, z, \lambda) &\rightarrow \vec{\Omega}(\vec{X}, \lambda), & z \rightarrow +\infty, \\ \vec{\varphi}_+(\vec{x}, z, \lambda) &\rightarrow \vec{s}_+(\vec{X}, \lambda), & z \rightarrow -\infty,\end{aligned}\tag{47}$$

and, consequently, the expression of $\vec{\varphi}_+$ in terms of $\vec{\varphi}_-$ is given by:

$$\vec{\varphi}_+(\vec{x}, z, \lambda) = \vec{s}_+(\vec{\varphi}_-(\vec{x}, z, \lambda), \lambda) = \vec{\varphi}_-(\vec{x}, z, \lambda) + \vec{\Delta}(\vec{\varphi}_-(\vec{x}, z, \lambda), \lambda).\tag{48}$$

To derive equations (47), it is sufficient to take the $z = \pm\infty$ limit of (27), using (44); equation (48) follows from the ring property and from (47).

The Jost solutions form a natural basis for expanding any eigenfunction of \hat{L}_1 . The expansion of ϕ^+ in terms of ϕ^- :

$$\phi^+(\vec{x}, z; \vec{\alpha}, \lambda) = \int_{\mathbb{R}^2} d\vec{\beta} S(\vec{\alpha}, \vec{\beta}, \lambda) \phi^-(\vec{x}, z; \vec{\beta}, \lambda)\tag{49}$$

defines the z -scattering datum S , which admits the standard integral representation:

$$S(\vec{\alpha}, \vec{\beta}, \lambda) = \delta(\vec{\alpha} - \vec{\beta}) + \int_{\mathbb{R}^3} d\vec{x} dz e^{-i\vec{\beta} \cdot \vec{X}} (\vec{u}_1(\vec{x}, z) \cdot \nabla_{\vec{x}}) \phi^+(\vec{x}, z; \vec{\alpha}, \lambda)\tag{50}$$

following from (30), where $\delta(\vec{\alpha} - \vec{\beta}) = \delta(\alpha_1 - \beta_1)\delta(\alpha_2 - \beta_2)$.

The z -scattering datum S exhibits the following Fourier representation in terms of the scattering vector $\vec{\Delta}$:

$$S(\vec{\alpha}, \vec{\beta}, \lambda) = \int_{\mathbb{R}^2} \frac{d\vec{\omega}}{(2\pi)^2} e^{i\vec{\omega} \cdot (\vec{\alpha} - \vec{\beta}) + i\vec{\alpha} \cdot \vec{\Delta}(\vec{\omega}, \lambda)}. \quad (51)$$

This basic formula is consequence of (28), (48) and (49). Indeed, substituting (28) $_{\pm}$ into (49), one obtains:

$$e^{i\vec{\alpha} \cdot \varphi_+} = \tilde{S}(\vec{\alpha}, \varphi_-, \lambda), \quad (52)$$

where \tilde{S} is the Fourier transform of S , with respect to $\vec{\beta}$:

$$\tilde{S}(\vec{\alpha}, \vec{\omega}, \lambda) = \int_{\mathbb{R}^2} d\vec{\beta} S(\vec{\alpha}, \vec{\beta}, \lambda) e^{i\vec{\beta} \cdot \vec{\omega}}. \quad (53)$$

Then, using (48), it follows that $\tilde{S}(\vec{\alpha}, \vec{\omega}, \lambda) = e^{i\vec{\alpha} \cdot \vec{s}_+(\vec{\omega}, \lambda)}$. Therefore the special dependence (28) of ϕ^{\pm} on $\vec{\alpha}$ only at the exponent implies that also the Fourier transform \tilde{S} of S depends on $\vec{\alpha}$ only at the exponent.

The direct problem is the mapping, via (50), from the two real potentials $u_1^{1,2}$, functions of the three real variables (\vec{x}, z) , to the scattering datum S , or, via (51), to the two real components $\Delta^{1,2}$ of $\vec{\Delta}$, functions of the three real variables $(\vec{\omega}, \lambda)$. Then, thanks to the ring property of the space of eigenfunctions, the counting is consistent.

The expansions of the analytic eigenfunctions ψ_{\pm} in terms of the Jost eigenfunctions:

$$\begin{aligned} \psi_+(\vec{x}, z; \vec{\alpha}, \lambda) &= \int_{\mathbb{R}^2} d\vec{\beta} K_+^{\pm}(\vec{\alpha}, \vec{\beta}, \lambda) \phi^{\pm}(\vec{x}, z; \vec{\beta}, \lambda), \\ \psi_-(\vec{x}, z; \vec{\alpha}, \lambda) &= \int_{\mathbb{R}^2} d\vec{\beta} K_-^{\pm}(\vec{\alpha}, \vec{\beta}, \lambda) \phi^{\pm}(\vec{x}, z; \vec{\beta}, \lambda) \end{aligned} \quad (54)$$

introduce the kernels $K_{\pm}^{\pm}(\vec{\alpha}, \vec{\beta}, \lambda)$; they are connected to S via the integral equations

$$K_{\pm}^-(\vec{\alpha}, \vec{\beta}, \lambda) = \int_{\mathbb{R}^2} d\vec{\gamma} K_{\pm}^+(\vec{\alpha}, \vec{\gamma}, \lambda) S(\vec{\gamma}, \vec{\beta}, \lambda), \quad (55)$$

which follow directly from (54), replacing ϕ^+ by its expansion (49) in terms of ϕ^- .

The reality of the vector fields and its consequence (42) imply the following reality constraints for the scattering data S and K (for real λ):

$$S^*(\vec{\alpha}, \vec{\beta}, \lambda) = S(-\vec{\alpha}, -\vec{\beta}, \lambda), \quad (56)$$

$$K_{\pm}^{\pm*}(\vec{\alpha}, \vec{\beta}, \lambda) = K_{\pm}^{\pm}(-\vec{\alpha}, -\vec{\beta}, \lambda). \quad (57)$$

Due to the special dependence (39) of ψ_{\pm} and ϕ_{\pm} on $\vec{\alpha}$, also the Fourier transforms $\tilde{K}_{\pm}^{\pm}(\vec{\alpha}, \vec{\omega}, \lambda)$ of $K_{\pm}^{\pm}(\vec{\alpha}, \vec{\beta}, \lambda)$ with respect to $\vec{\beta}$, depend on $\vec{\alpha}$ only at the exponents:

$$\tilde{K}_{\pm}^{\pm}(\vec{\alpha}, \vec{\omega}, \lambda) = \left(\tilde{K}_{\pm}^{\pm 1}(\vec{\omega}, \lambda) \right)^{\alpha_1} \left(\tilde{K}_{\pm}^{\pm 2}(\vec{\omega}, \lambda) \right)^{\alpha_2}, \quad (58)$$

yielding the following Fourier representations of the kernels K_{\pm}^{\pm} :

$$\begin{aligned} K_{+}^{\pm}(\vec{\alpha}, \vec{\beta}, \lambda) &= \int_{\mathbb{R}^2} \frac{d\vec{\omega}}{(2\pi)^2} e^{i\vec{\omega} \cdot (\vec{\alpha} - \vec{\beta})} e^{\vec{\alpha} \cdot \vec{\chi}_{+}^{\mp}(\vec{\omega}, \lambda)}, \\ K_{-}^{\pm}(\vec{\alpha}, \vec{\beta}, \lambda) &= \int_{\mathbb{R}^2} \frac{d\vec{\omega}}{(2\pi)^2} e^{i\vec{\omega} \cdot (\vec{\alpha} - \vec{\beta})} e^{\vec{\alpha} \cdot \vec{\chi}_{-}^{\pm}(\vec{\omega}, \lambda)}, \end{aligned} \quad (59)$$

where

$$\vec{\chi}_{\pm}^{\mp} = (\chi_{\pm}^{\mp 1}, \chi_{\pm}^{\mp 2}), \quad \chi_{\pm}^{\mp j} = \ln \tilde{K}_{\pm}^{\pm j} - i\omega_j, \quad j = 1, 2, \quad (60)$$

and yielding also the representations of the analytic eigenfunctions $\vec{\pi}_{\pm}$ in terms of $\vec{\varphi}_{\pm}$ and $\vec{\chi}_{\pm}^{\mp}$:

$$\begin{aligned} \vec{\pi}_{+}(\vec{x}, z, \lambda) &= i\vec{\varphi}_{+}(\vec{x}, z, \lambda) + \vec{\chi}_{+}^{\mp}(\vec{\varphi}_{+}(\vec{x}, z, \lambda), \lambda), \\ \vec{\pi}_{-}(\vec{x}, z, \lambda) &= i\vec{\varphi}_{-}(\vec{x}, z, \lambda) + \vec{\chi}_{-}^{\pm}(\vec{\varphi}_{-}(\vec{x}, z, \lambda), \lambda). \end{aligned} \quad (61)$$

The proof is the same as before; replacing the exponential representations (28) and (39b) of ϕ^{\pm} and ψ_{\pm} , f.i., in (54a), one obtains the relation

$$e^{\vec{\alpha} \cdot \vec{\pi}_{+}} = \tilde{K}_{+}^{\pm}(\vec{\alpha}, \vec{\varphi}_{+}, \lambda), \quad (62)$$

proving that the dependence of \tilde{K}_{+}^{\pm} on $\vec{\alpha}$ is only at the exponent. In addition, using the expression of \tilde{K}_{\pm}^{\pm} in terms of $\vec{\chi}_{\pm}^{\mp}$, one obtains (61a).

We have clarified the impact of the ring property on the kernels K_{\pm}^{\pm} . The analyticity properties of the $z = \pm\infty$ asymptotics of ψ_{\pm} , discussed in the previous section, imply instead the following triangular structures:

$$\begin{aligned} K_{+}^{\pm}(\vec{\alpha}, \vec{\beta}, \lambda) &= \delta(\vec{\alpha} - \vec{\beta}) + \theta(\pm(\alpha_1 - \beta_1)) \check{K}_{+}^{\pm}(\vec{\alpha}, \vec{\beta}, \lambda), \\ K_{-}^{\pm}(\vec{\alpha}, \vec{\beta}, \lambda) &= \delta(\vec{\alpha} - \vec{\beta}) + \theta(\mp(\alpha_1 - \beta_1)) \check{K}_{-}^{\pm}(\vec{\alpha}, \vec{\beta}, \lambda) \end{aligned} \quad (63)$$

(the same features were observed in [12]). Therefore the linear integral equation (55) describes a factorization problem allowing to construct, in principle, the kernels K_{\pm}^{\pm} in terms of the given scattering datum S .

Comparing (63) and (58), one infers that \tilde{K}_{+}^{-} , \tilde{K}_{-}^{+} are analytic in the upper half ω^1 - plane and \tilde{K}_{+}^{+} , \tilde{K}_{-}^{-} are analytic in the lower half ω^1 - plane.

Since these analyticity properties are valid $\forall \vec{\alpha} \in \mathbb{R}^2$, it follows that $\tilde{K}_{\pm}^{\pm 1}$, $\tilde{K}_{\pm}^{\pm 2}$ in (58) cannot have poles and zeroes in their analyticity domains. Therefore also $\chi_{\pm}^{\pm 1,2}(\vec{\omega}, \lambda)$ and $\chi_{\pm}^{\mp 1,2}(\vec{\omega}, \lambda)$ are analytic, respectively, in the upper and lower half ω_1 - plane, with asymptotics:

$$\chi_{\pm}^{\pm 1,2}(\vec{\omega}, \lambda) \rightarrow 0, \quad |\vec{\omega}|^2 \rightarrow \infty, \quad \lambda \neq 0. \quad (64)$$

At last, plugging the Fourier representations (51) and (59) into the integral equation (55), one obtains the following basic equations:

$$\begin{aligned} \vec{\chi}_+^+(\vec{\omega}, \lambda) - \vec{\chi}_+(\vec{\omega} + \vec{\Delta}(\vec{\omega}, \lambda), \lambda) &= i\vec{\Delta}^{1,2}(\vec{\omega}, \lambda), \\ \vec{\chi}_-^-(\vec{\omega}, \lambda) - \vec{\chi}_-^+(\vec{\omega} + \vec{\Delta}(\vec{\omega}, \lambda), \lambda) &= i\vec{\Delta}^{1,2}(\vec{\omega}, \lambda), \quad \vec{\omega} \in \mathbb{R}^2, \quad \vec{\Delta} \in \mathbb{R}^2, \quad \lambda \in \mathbb{R}, \end{aligned} \quad (65)$$

wich must be viewed as “inhomogeneous Riemann - Hilbert problems on the line $\text{Im } \omega^1 = 0$, with the shift $\Delta^1(\vec{\omega}, \lambda)$ ” [13, 14] (the real variables ω^2 and λ appear just as parameters). Given $\vec{\Delta}(\vec{\omega}, \lambda)$, namely, given the shift and the inhomogeneity in (65), one can construct, in principle, the unique solutions $\vec{\chi}_{\pm}^{\pm}$ of these Riemann - Hilbert problems, expressed in terms of the following linear integral equations of Fredholm type :

$$\begin{aligned} \vec{\chi}_+^+(\omega^1, \omega^2, \lambda) + \frac{1}{2\pi i} \int_{\mathbb{R}} d\tau M(\omega^1, \tau, \lambda) \vec{\chi}_+^+(\tau, \omega^2, \lambda) + \vec{L}(\omega^1, \omega^2, \lambda) &= \vec{0}, \\ \vec{\chi}_-^-(\omega^1, \omega^2, \lambda) - \frac{1}{2\pi i} \int_{\mathbb{R}} d\tau M(\omega^1, \tau, \lambda) \vec{\chi}_-^-(\tau, \omega^2, \lambda) + \vec{L}(\omega^1, \omega^2, \lambda) &= \vec{0}, \end{aligned} \quad (66)$$

where

$$\begin{aligned} M(\omega^1, \tau, \omega^2, \lambda) &= \frac{\partial s_+^1(\tau, \omega^2, \lambda)/\partial \tau}{s_+^1(\tau, \omega^2, \lambda) - s_+^1(\omega^1, \omega^2, \lambda)} - \frac{1}{\tau - \omega^1}, \\ \vec{L}(\omega^1, \omega^2, \lambda) &= -i\vec{\Delta}(\omega^1, \omega^2, \lambda) + \frac{1}{2\pi} P \int_{\mathbb{R}} d\tau \frac{\partial s_+^1(\tau, \omega^2, \lambda)/\partial \tau}{s_+^1(\tau, \omega^2, \lambda) - s_+^1(\omega^1, \omega^2, \lambda)} \vec{\Delta}(\tau, \omega^2, \lambda) \\ s_+^1(\omega^1, \omega^2, \lambda) &= \omega^1 + \Delta^1(\omega^1, \omega^2, \lambda). \end{aligned} \quad (67)$$

The inverse problem. Once $\vec{\chi}_{\pm}^{\pm}$ are known, one reconstructs the eigenfunction $\vec{\varphi}_-$ solving the following nonlinear integral equation

$$i\vec{\varphi}_-(\vec{x}, z, \lambda) + \hat{P}_{\lambda}^- \vec{\chi}_+^+(\vec{\varphi}_-(\vec{x}, z, \lambda), \lambda) + \hat{P}_{\lambda}^+ \vec{\chi}_-^-(\vec{\varphi}_-(\vec{x}, z, \lambda), \lambda) = i\vec{X}, \quad (68)$$

where \hat{P}_{λ}^{\pm} are respectively the (+) and (-) analyticity projectors with respect to λ :

$$\hat{P}_{\lambda}^{\pm} = \pm \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{d\lambda'}{\lambda' - \lambda \mp i\epsilon}. \quad (69)$$

Equation (68) is obtained applying \hat{P}_λ^- and \hat{P}_λ^+ respectively to the analytic (and decreasing at $\lambda = \infty$) expressions $(\vec{\pi}_+ - i\vec{X})$ and $(\vec{\pi}_- - i\vec{X})$, subtracting the resulting equations, and using (61).

From the solution of (68), one reconstruct the potentials \vec{u}_1 using the formulas

$$\vec{u}_1(\vec{x}, z) = i\partial_x \lim_{\lambda \rightarrow \infty} \left(\lambda [i(\vec{\varphi}_-(\vec{x}, z, \lambda) - \vec{X}) + \vec{\chi}_+^+(\vec{\varphi}_-(\vec{x}, z, \lambda), \lambda)] \right), \quad (70)$$

consequence of (36) and (41).

Therefore we have constructed an IST which involves a nonlinear step: the nonlinear equation (68) of the inverse problem. The above results can be summarized as follows.

Nonlinear IST scheme. The direct scattering is the mapping from the potential $\vec{u}_1(\vec{x}, z)$ to the scattering vector $\vec{\Delta}(\vec{\omega}, \lambda)$. Knowing $\vec{\Delta}(\vec{\omega}, \lambda)$, one constructs the vectors $\vec{\chi}_+^+(\vec{\omega}, \lambda)$ and $\vec{\chi}_-^-(\vec{\omega}, \lambda)$ solving the linear RH problems with shift (65). In the inverse problem, from the knowledge of $\vec{\chi}_+^+(\vec{\omega}, \lambda)$, $\vec{\chi}_-^-(\vec{\omega}, \lambda)$, one solves the nonlinear integral equation (68) for $\vec{\varphi}_-(\vec{x}, z, \lambda)$ and, finally, one reconstructs the potentials $\vec{u}_1(\vec{x}, z)$ from (70).

It is quite remarkable that the inverse problem becomes linear, if expressed in terms of the Jost and analytic eigenfunctions ϕ^\pm and ψ_\pm . Indeed, from the solutions $\vec{\chi}_\pm^\pm$ of the RH problems (65) one can construct, via (59), the factorization kernels K_\pm^\pm . Known K_+^-, K_-^+ , one reconstructs the Jost eigenfunction ϕ^- by solving the following *linear* integral equation:

$$\begin{aligned} \phi^-(\vec{\alpha}, \lambda) e^{-i\vec{\alpha} \cdot \vec{X}} + \hat{P}_\lambda^+ \int_{\mathbb{R}^2} d\vec{\beta} \theta(-(\alpha_1 - \beta_1)) \check{K}_-^-(\vec{\alpha}, \vec{\beta}, \lambda) \phi^-(\vec{\beta}, \lambda) e^{-i\vec{\alpha} \cdot \vec{X}} + \\ \hat{P}_\lambda^- \int_{\mathbb{R}^2} d\vec{\beta} \theta(-(\alpha_1 - \beta_1)) \check{K}_+^-(\vec{\alpha}, \vec{\beta}, \lambda) \phi^-(\vec{\beta}, \lambda) e^{-i\vec{\alpha} \cdot \vec{X}} = 1. \end{aligned} \quad (71)$$

This equation, in which we have omitted, for simplicity, the parametric dependence on (\vec{x}, z) , is consequence of the analyticity properties of ψ_\pm , and is obtained multiplying the equations

$$\begin{aligned} \psi_+(\vec{x}, z; \vec{\alpha}, \lambda) &= \int_{\mathbb{R}^2} d\vec{\beta} K_+^-(\vec{\alpha}, \vec{\beta}, \lambda) \phi^-(\vec{x}, z; \vec{\beta}, \lambda), \\ \psi_-(\vec{x}, z; \vec{\alpha}, \lambda) &= \int_{\mathbb{R}^2} d\vec{\beta} K_-^-(\vec{\alpha}, \vec{\beta}, \lambda) \phi^-(\vec{x}, z; \vec{\beta}, \lambda) \end{aligned} \quad (72)$$

in (54) by $e^{-i\vec{\alpha} \cdot \vec{X}}$, subtracting 1, applying respectively \hat{P}_λ^- and \hat{P}_λ^+ , and adding the resulting equations. This inversion procedure has been already presented in [15].

Once ϕ^\pm are known and, via (72), ψ^\pm are also known, the potential \vec{u}_1 is reconstructed, via (34) and (36), through the formulae:

$$\begin{aligned} u_1^1(\vec{x}, z) &= i\partial_x \lim_{\lambda \rightarrow \infty} \lambda \left(\psi_\pm e^{-i\vec{\alpha} \cdot \vec{X}} - 1 \right) \Big|_{\vec{\alpha}=(1,0)}, \\ u_1^2(\vec{x}, z) &= i\partial_x \lim_{\lambda \rightarrow \infty} \lambda \left(\psi_\pm e^{-i\vec{\alpha} \cdot \vec{X}} - 1 \right) \Big|_{\vec{\alpha}=(0,1)}. \end{aligned} \quad (73)$$

This second version of the IST consists of linear steps only, and can be summarized as follows.

Linear IST scheme. The direct scattering is the mapping from the potential $\vec{u}_1(\vec{x}, z)$ to the scattering vector $\vec{\Delta}(\vec{\omega}, \lambda)$. Knowing $\vec{\Delta}(\vec{\omega}, \lambda)$, one constructs the vectors $\vec{\chi}_+^+(\vec{\omega}, \lambda)$ and $\vec{\chi}_-^-(\vec{\omega}, \lambda)$ solving the linear RH problems with shift (65). Then, using the Fourier representations (59), one obtains the kernels K_-^+ and K_+^- . In the inverse problem, the Jost eigenfunction is then reconstructed through the linear integral equation (71), the analytic eigenfunctions via (72), and the potentials via (73).

The small field limit. In the small field limit $|u_1^{1,2}| = O(\epsilon) \ll 1$, the scattering vector $\vec{\Delta}$ is expressed in terms of \vec{u}_1 in the following way:

$$\vec{\Delta}(\vec{\omega}, \lambda) = \int_{\mathbb{R}} dz \vec{u}_1(\omega^1 + \lambda z, \omega^2, z) + O(\epsilon^2), \quad (74)$$

while the Riemann - Hilbert problems with shift (65) reduce to the scalar Riemann - Hilbert problems:

$$\vec{\chi}_+^+(\vec{\omega}, \lambda) - \vec{\chi}_+^-(\vec{\omega}, \lambda) = \vec{\chi}_-^-(\vec{\omega}, \lambda) - \vec{\chi}_-^+(\vec{\omega}, \lambda) = i\vec{\Delta}(\vec{\omega}, \lambda) + O(\epsilon^2), \quad (75)$$

entailing

$$\begin{aligned} \vec{\chi}_+^+(\vec{\omega}, \lambda) &= \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{d\xi dz}{\xi - (\omega_1 \pm i0)} \vec{u}_1(\xi + \lambda z, \omega_2, z) + O(\epsilon^2), \\ \vec{\chi}_-^-(\vec{\omega}, \lambda) &= -\frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{d\xi dz}{\xi - (\omega_1 \pm i0)} \vec{u}_1(\xi + \lambda z, \omega_2, z) + O(\epsilon^2). \end{aligned} \quad (76)$$

At last, the small field limit of the inverse problem equations (71) and (68) reads

$$i(\vec{\varphi}_-(\vec{x}, z, \lambda) - \vec{X}) + \hat{P}_\lambda^+ \vec{\chi}_-^-(\vec{X}, \lambda) + \hat{P}_\lambda^- \vec{\chi}_+^+(\vec{X}, \lambda) = O(\epsilon^2). \quad (77)$$

The t-evolution. The time evolution is introduced in the usual way. We observe that the Jost eigenfunction ϕ^+ solves the equation $\hat{L}_2\phi^+ = i\lambda\alpha_2\phi^+$ which, evaluated at $z = -\infty$ using also (49), leads to

$$S(\vec{\alpha}, \vec{\beta}, \lambda, t) = S(\vec{\alpha}, \vec{\beta}, \lambda, 0)e^{i\lambda(\alpha_2 - \beta_2)t}, \quad (78)$$

From (78) and (55) one infers the same elementary t -dependence for K_{\pm}^{\pm} :

$$K_{\pm}^{\pm}(\vec{\alpha}, \vec{\beta}, \lambda, t) = K_{\pm}^{\pm}(\vec{\alpha}, \vec{\beta}, \lambda, 0)e^{i\lambda(\alpha_2 - \beta_2)t}. \quad (79)$$

At last, using (78) and the Fourier representation (51) of S , one obtains the t -evolution of the scattering vector $\vec{\Delta}$:

$$\vec{\Delta}(\vec{\omega}, \lambda, t) = \vec{\Delta}(\vec{\omega} - (0, \lambda)t, \lambda). \quad (80)$$

The heavenly reduction. The heavenly equation corresponds to the Hamiltonian reduction (10). In this case, if $f_{1,2}$ are solutions of $\hat{L}_i f_{1,2} = 0$, then the Poisson bracket (15) of $f_{1,2}$ is also a solution: $\hat{L}_i(\{f_1, f_2\}_{\vec{x}}) = 0$. Therefore the solution space of the equation $\hat{L}_1 f = 0$ is not only a ring, but also a Lie algebra, with Lie bracket given by the Poisson bracket (15). This result follows immediately from equation (14a) and from the Jacobi identity.

Under the hamiltonian reduction (10), equation (26) gets the form

$$\frac{d\vec{r}}{dz} = \{\vec{r}, H_1 + \lambda r_2\}_{\vec{r}}. \quad (81)$$

Thus the z -evolution of \vec{r} is canonical, i.e. it preserves the Poisson brackets of $r_1(z)$ and $r_2(z)$. So, comparing eqs.(26b) and (44), one obtains the following constraint for the scattering vector $\vec{\Delta}$:

$$\{s_+^1, s_+^2\}_{\vec{\omega}} = \left| \frac{\partial(s_+^1, s_+^2)}{\partial(\omega^1, \omega^2)} \right| = 1. \quad (82)$$

3 Conclusions and open problems

In this paper we have constructed the IST for multidimensional vector fields, and we have used it to solve the Cauchy problem for the heavenly equation. Although these results have been derived in the particular case (11), directly

connected to the heavenly equation, they can be extended in a straightforward way to the general vector fields (2) and to the $(4 + N)$ dimensional PDEs (5).

Interesting open problems under present investigation are: i) the transition from the above formal results to rigorous ones. ii) The study of the spectral mechanisms (if any) which could cause a breaking of the initial profile for the PDEs under investigation (a typical feature of quasi-linear PDEs). iii) The construction of explicit solutions within this spectral formalism.

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